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Engineering Notes

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Nonlinear Optimal Control Analysis Using State-Dependent Matrix Exponential and Its Integrals

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Introduction

THERE exist various strategies [1,2] for reaching numerical solutions to the nonlinear optimal control problem. The direct-multiple-shooting (DMS) method is usually preferred for analyzing general nonlinear optimal control problems due to its convenience in handling system constraints and a large convergence radius compared with other methods. However, these advantages can be decreased in case the estimation of the related Karush–Kuhn–Tucker (KKT) system is not accurate enough to guarantee robust analyses. The related estimation errors are originated from time integration and finite difference approximation in the standard DMS method.

The present work intends to propose a new method of accurately estimating the KKT system. For this purpose, the state-dependent coefficient (SDC) factorization method, which has been successfully implemented in the state-dependent Riccati equation (SDRE) technique [3–5], is used to derive a linear system structure from the nonlinear motion equation. Applying the linear system theory to the resultant SDC form of equations, the KKT system can be built without resorting to any time integration and finite difference formula to calculate of gradients and Hessian matrices, as opposed to the standard DMS method. The present paper proves that the convergence and the accuracy of the DMS method can be greatly enhanced through the applications of the new method to the rotorcraft trajectory tracking problem.

Direct-Multiple-Shooting Framework for Linear Quadratic Tracking Problems

Nonlinear optimal control problems over $t \in [t_0, t_f]$ can be represented by the standard Bolza form:

$$\min_{x,u,t_f} J(x,u,t_f) = \phi(x(t_f)) + \int_{t_0}^{t_f} \Phi(x,u)\mathrm{d},t$$
$$\text{subject to } \dot{x}(t) = f(x(t),u(t),t), \qquad t \in [t_0,t_f] \qquad x(t_0) = x_0$$
$$h(x(t_f),t_f) = 0, \qquad g(x(t),u(t),t) \le 0, \qquad t \in [t_0,t_f] \tag{1}$$

If time horizon $[t_0, t_f]$ is divided by N-shooting intervals as $[t_j, t_{j+1}]$ $(j = 0,\ldots,N-1)$, the related nonlinear programming (NLP) problem in the DMS framework can be formulated in finite dimension to find initial states $x(t_j) = s_j$ at each jth node and piecewise-constant controls $u(t) = q_j$ over $t \in [t_j, t_{j+1})$ as

$$\min J(\mathbf{s},\mathbf{q}) = \phi(s_N) + \sum_{i=0}^{N-1} J_j(s_j,q_j) \quad \text{with}$$
$$J_j(s_j,q_j) = \int_{t_j}^{t_{j+1}} \Phi(x_j(t;s_j,q_j),q_j)\mathrm{d}t \tag{2}$$

$$\text{subject to } h_j(s_j,s_{j+1},q_j) = s_j + \int_{t_j}^{t_{j+1}} f(x_j(t),q_j,t)\mathrm{d}t - s_{j+1} = 0$$
$$j = 0,\ldots,N-1 \tag{3}$$

$$h_N(s_N,t_N) = h(s_N,t_N) = 0 \tag{4}$$

$$g_j(s_j,q_j,t_j) \le 0, \qquad j = 1,\ldots,N \tag{5}$$

The preceding NLP problem is efficiently solved using the sequential quadratic programming method. The detailed derivation of the corresponding KKT system can be found in [6], and the results for $j = 2,\ldots,N-1$ can be summarized as

$$\begin{bmatrix} H_j^{ss} & (H_j^{qs})^T & (F_j)^T & (G_j)^T \\ H_j^{qs} & H_j^{qq} & (D_j)^T & (E_j)^T \\ F_j & D_j & & \\ G_j & E_j & & \end{bmatrix} \begin{bmatrix} p_j^s \\ p_j^q \\ \lambda_j \\ \mu_j \end{bmatrix} + \begin{bmatrix} (J_j^s)^T - \lambda_{j-1} \\ (J_j^q)^T \\ h_j - p_{j+1}^s \\ g_j \end{bmatrix} = 0$$

where p_j^s and p_j^q are search directions for state and control variables, J_j^s and J_j^q are gradients of cost function with respect to state and control variables, and λ_j and μ_j are Lagrange multipliers for constraints.

In the preceding equation, only active inequality constraints $g_j > 0$ with $\mu_j \ge 0$ should be included in the KKT system, and its gradients G_j, E_j can be estimated using the finite difference formula. The present method focuses on the way of building Hessian matrices H_j, gradients J_j of the cost function, and Jacobians F_j and D_j of equality constraints $h_j = 0\,(j = 0,\ldots,N-1)$. The standard method of estimating these vectors and matrices requires time integrations of Eq. (3).

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Consider the following cost function for a linear quadratic (LQ) tracking problem with a prescribed reference trajectory $x_R(t)$ and weighting matrices of $Q_C \geq 0$ and $R > 0$:

$$\Phi_j(s_j, q_j, t) = (x(t) - x_R(t))^T Q_C (x(t) - x_R(t)) + q_j^T R q_j \tag{6}$$

The solution of a linear system shown in Eq. (7) has the closed-form expression of Eq. (8) over the time interval $t \in [t_j, t_{j+1})$:

$$\dot{x}(t) = Ax(t) + Bq_j \tag{7}$$

where $x(t_j) = s_j$,

$$x(t) = F(\Delta)s_j + H(\Delta)q_j \tag{8}$$

where

$$F(\Delta) = e^{A\Delta} \tag{9}$$

$$H(\Delta) = \int_{t_j}^{t} e^{A(t-\tau)} B \mathrm{d}\tau, \qquad \Delta = t - t_j \tag{10}$$

If we replace $x_R(t)$ with its mean value $\bar{x}_{R,j}$ over $t \in [t_j, t_{j+1}]$, the local cost function can be approximated as

$$J_j(s_j, q_j) \approx \frac{1}{2} s_j^T Q s_j + s_j^T M q_j + \frac{1}{2} q_j^T W q_j + \frac{\Delta_j}{2} q_j^T R q_j - \bar{x}_{R,j}^T Q_C (H_{B=I}) s_j - \bar{x}_{R,j}^T (M_{F=I}) q_j + \frac{\Delta_j}{2} \bar{x}_{R,j}^T Q_C \bar{x}_{R,j}$$

where

$$Q = \int_0^{\Delta_j} F(\tau)^T Q_C F(\tau) \mathrm{d}\tau \tag{11}$$

$$M = \int_0^{\Delta_j} F(\tau)^T Q_C H(\tau) \mathrm{d}\tau \tag{12}$$

$$W = \int_0^{\Delta_j} H(\tau)^T Q_C H(\tau) \mathrm{d}\tau \tag{13}$$

$$H_{B=I} = \int_0^{\Delta_j} e^{A(\Delta_j - \tau)} \mathrm{d}\tau \tag{14}$$

$$M_{F=I} = \int_0^{\Delta_j} Q_C H(\tau) \mathrm{d}\tau \tag{15}$$

Therefore, the gradients and the Hessians of the local cost function can be written as

$$\begin{aligned} J_j^s &= \frac{\partial J_j}{\partial s_j} = s_j^T Q + q_j^T M^T - \bar{x}_{R,j}^T Q_C (H_{B=I}) \\ J_j^q &= \frac{\partial J_j}{\partial q_j} = s_j^T M + q_j^T W + \Delta_j q_j^T R - \bar{x}_{R,j}^T (M_{F=I}) \end{aligned} \tag{16}$$

$$\begin{aligned} H_j^{ss} &= \frac{\partial^2 J_j}{\partial s_j^2} = Q, \qquad H_j^{qs} = \frac{\partial^2 J_j}{\partial q_j \partial s_j} = M^T \\ H_j^{qq} &= \frac{\partial^2 J_j}{\partial q_j^2} = W^T + \Delta_j R \end{aligned} \tag{17}$$

Using the state solution $x(t_{j+1})$, the Jacobians of the equality constraint can be written as

$$F_j = \frac{\partial h_j}{\partial s_j} = F(\Delta_j), \qquad D_j = \frac{\partial h_j}{\partial q_j} = H(\Delta_j) \tag{18}$$

Therefore, the KKT system has been built using the matrix exponential and its weighted integrals. In this study, the fast and accurate algorithm by Loan [7] is used to simultaneously calculate the matrix exponential and weighted integrals shown in Eqs. (9–15).

Formulation for Nonlinear Optimal Control Problems

The KKT system for nonlinear systems can be built as in the LQ tracking problem by using the following SDC form of motion equations, which presents the linear-system-like structure with state-dependent coefficient matrices:

$$\dot{x} = A(x)x + B(x)u$$

A general nonlinear system $\dot{x}(t) = f(x, u)$ can be transformed into the SDC form only if it has a conforming structure [3–5] that satisfies $f(x, u) = 0$ at $x = u = 0$. To drive the conforming structure, consider the following motion equations at an arbitrary time instance and at an equilibrium condition:

$$\dot{x} = F(x, u) \tag{19}$$

$$\dot{x}_0 = F(x_0, u_0) \tag{20}$$

where x_0 and u_0 denote equilibrium states and controls, respectively.

If the variation of states and controls is denoted with $\tilde{x}$ and $\tilde{u}$, the motion equation for $\tilde{x}$ can be written as

$$x = x_0 + \tilde{x}, \qquad u = u_0 + \tilde{u}$$

$$\dot{\tilde{x}} = \tilde{F}(\tilde{x}, \tilde{u}) = F(x, u) - F(x_0, u_0) = A(\tilde{x})\tilde{x} + B(\tilde{x})\tilde{u} \tag{21}$$

The control derivative matrix $B(\tilde{x})$ can be estimated by applying a finite difference formula to the following equation:

$$B(\tilde{x}) = \frac{\partial \tilde{F}}{\partial \tilde{u}}$$

The state derivative matrix $A(\tilde{x})$ can be computed using its least-squares solution [5] around $\tilde{x} \in R^N$ of the following equation:

$$A(\tilde{x})\tilde{x} = \tilde{F}(\tilde{x}, \tilde{u}) - B(\tilde{x})\tilde{u}$$

The initial value problem to get the solution of Eq. (3) can be defined using the following SDC form, the state x_0, and the control u_0 at an equilibrium condition as

$$\begin{aligned} &\dot{\tilde{x}}(t) = A_j(\tilde{x})\tilde{x}(t) + B_j(\tilde{x})\tilde{q}_j, \\ &t \in [t_j, t_{j+1}), j = 0, 1, \ldots, N-1, \qquad \tilde{x}(t_j) = s_j - x_0, \\ &\tilde{q}_j = q_j - u_0 \end{aligned} \tag{22}$$

The corresponding vectors and matrices in the KKT system can be simultaneously obtained by assuming that the coefficient matrices $A_j(\tilde{x})$ and $B_j(\tilde{x})$ are locally constant as in the SDRE technique [3–5] and by using the present matrix exponential approach. The solution of Eq. (22) can be approximated by

$$x(t) = x_0 + F(\Delta)\tilde{s}_j + H(\Delta)\tilde{q}_j$$

where $\tilde{s}_j = s_j - x_0$. Using this solution, the gradients of the quadratic cost function can be represented as

$$J_j^s = \tilde{s}_j^T Q + \tilde{q}_j^T M^T + (x_0 - \bar{x}_{R,j})^T Q_C (H_{B=I}) \tag{23}$$

$$J_j^q = \tilde{s}_j^T M + \tilde{q}_j^T W + h_j \tilde{q}_j^T R + (x_0 - \bar{x}_{R,j})^T (M_{F=I}) \tag{24}$$

Therefore, Jacobians F_j and D_j of the equality constraints and Hessian matrices H_j have the same form as Eqs. (17) and (18).

Table 1 Codes for the identification of calculation methods

Items in KKT system	Estimation methods	Code
Functions of the cost function and continuity conditions	Time integration	F0
	SDME	F1
Gradients of the cost function and continuity conditions	Finite difference formula	G0
	SDME	G1
Hessian matrices for the cost function	Finite difference formula	H0
	SDME	H1
	BFGS with initial Hessian matrix by using	
	Identity matrix	H3
	Finite difference formula	H4
	SDME	H5

However, the gradients J_j^s and J_j^q of the cost function contain additional terms due to the equilibrium states. Because various numerical experiments with Eqs. (23) and (24) present divergent solutions, the following modified equations for local gradients are used, with which the trajectory tracking capability can be retained:

$$J_j^s = \tilde{s}_j^T Q + \tilde{q}_j^T M^T + \Delta\bar{s}_j^T Q_C (H_{B=I}) \tag{25}$$

$$J_j^q = \tilde{s}_j^T M + \tilde{q}_j^T W + h_j \tilde{q}_j^T R + \Delta\bar{s}_j^T (M_{F=I}) \tag{26}$$

where $\Delta\bar{s}_j$ represents the trajectory deviation at the middle of the shooting interval $[t_j, t_{j+1}]$ and is approximated by

$$\Delta\bar{s}_j^T = \tfrac{1}{2}(s_j + s_{j+1}) - x_R(\bar{t}_j) \tag{27}$$

where $\bar{t}_j = 0.5(t_j + t_{j+1})$. Because the matrix exponential and its integrals in the preceding formulation are state-dependent, these matrices should be calculated at each shooting interval. Hereinafter, the abbreviation SDME is used to represent the present method of building the KKT system using the state-dependent matrix exponential and its integrals.

Applications and Discussions

The numerical methods outlined in the previous sections are applied to an optimal control formulation for a slalom maneuver of the BO-105 helicopter and modeling details are covered in [8]. The lateral-position change during the slalom maneuver is prescribed with the following formula with a maneuver entry time of $t_{\text{entry}} = 1.0$ s and an exit time of $t_{\text{finish}} = 9.0$ s:

$$\Delta y(\bar{t}) = \frac{5}{46.8}[32 + \sin(2\pi\bar{t}) - 20\sin(4\pi\bar{t}) + 2\sin(8\pi\bar{t})] \text{ m}$$

where

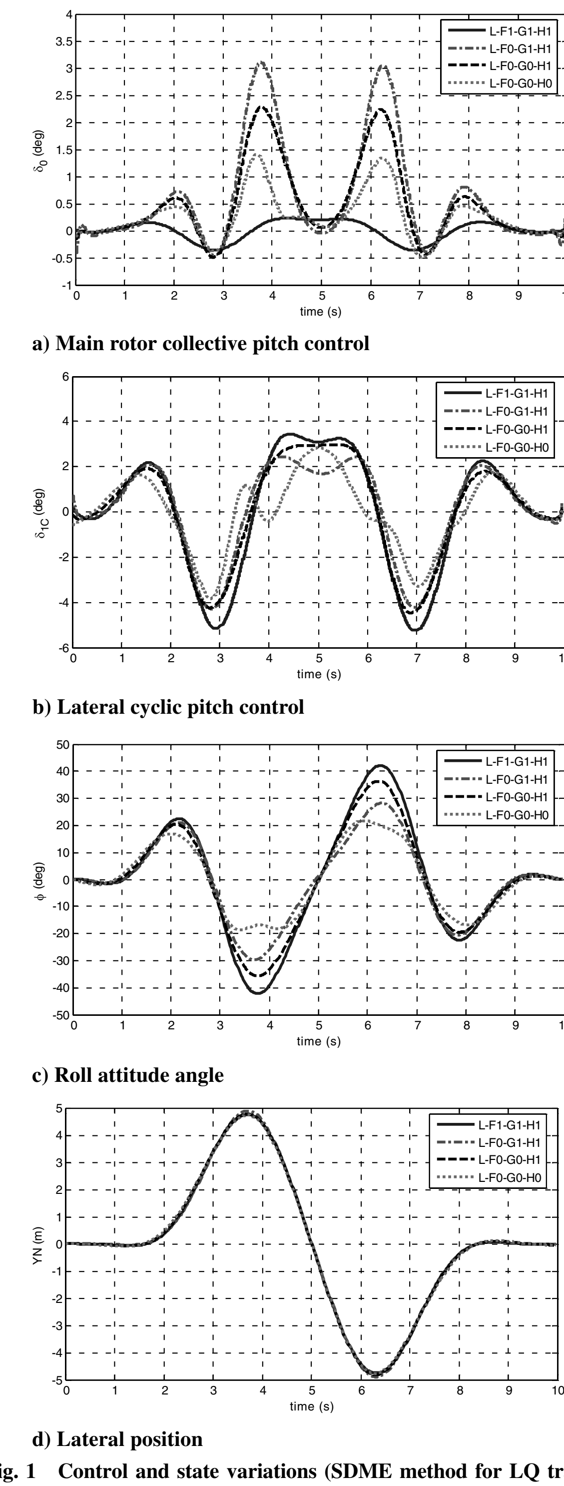
$$\bar{t} = \frac{t - t_{\text{entry}}}{t_{\text{finish}} - t_{\text{entry}}} \qquad 0 \le \bar{t} \le 1$$

The other target states are set to be the trim states. The quadratic cost function is defined with the following control and state weighting matrices, and no inequality constraints are imposed, for simplicity of comparison:

$$R = \text{diag}(50, 50, 50, 50)$$

$$Q_C = \text{diag}(1, 1, 1, 0.4, 0.2, 2, 0.1, 0.1, 50, 0, 5, 100)$$

Shooting nodes are evenly distributed over $t \in [0, 10]$ s with numbers of 500 for linear systems and 800 for nonlinear systems. When time integration is required, the 4-stage Runge–Kutta method was used with 32 time intervals at each shooting interval. The major features of the present method were investigated by solving the LQ tracking problems to differentiate the pros and cons of various approaches. Based upon these results, the SDME technique was applied to the nonlinear optimal control problem. Table 1 presents a summary of the possible methods of building each component of the KKT system, along with the corresponding identification codes. If



a) Main rotor collective pitch control

b) Lateral cyclic pitch control

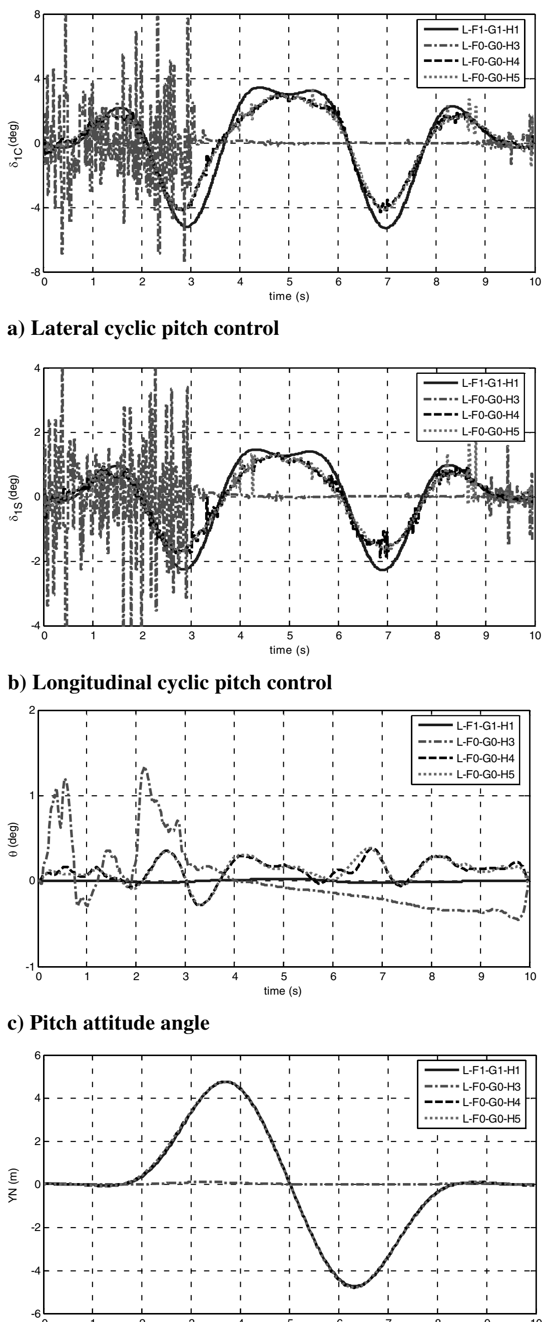
c) Roll attitude angle

d) Lateral position

Fig. 1 Control and state variations (SDME method for LQ tracking problem).

the linear system and the nonlinear system are represented by L and N, respectively, each method can be classified by combining 4 different codes. As an example, the code N-F0-G1-H3 represents the nonlinear analysis with a time integrator, the SDME method for gradients, and the iterative Broydon–Fletcher–Goldfrab–Shannon (BFGS) formula for Hessian update, initialized with an identity matrix.

Figure 1 shows the time history of controls and states for the LQ tracking problem. The solutions significantly depend on the ways of estimating the KKT system matrices. Because the matrix exponential approach provides a nearly exact computation of function values, gradients, and Hessian matrices, the root causes of any deviation from L-F1-G1-H1 can be considered to be errors in time integration and in finite difference formula used in each method. Figure 2 compares the effect of various initialization methods of the iterative BFGS formula for Hessian matrices. Solutions based on the L-F0-G0-H4 or the L-F0-G0-H5 methods present highly oscillatory behaviors, which show that numerical error in the initial guesses of the Hessian matrices seriously affects the quality of the updated Hessian matrices using the BFGS method.



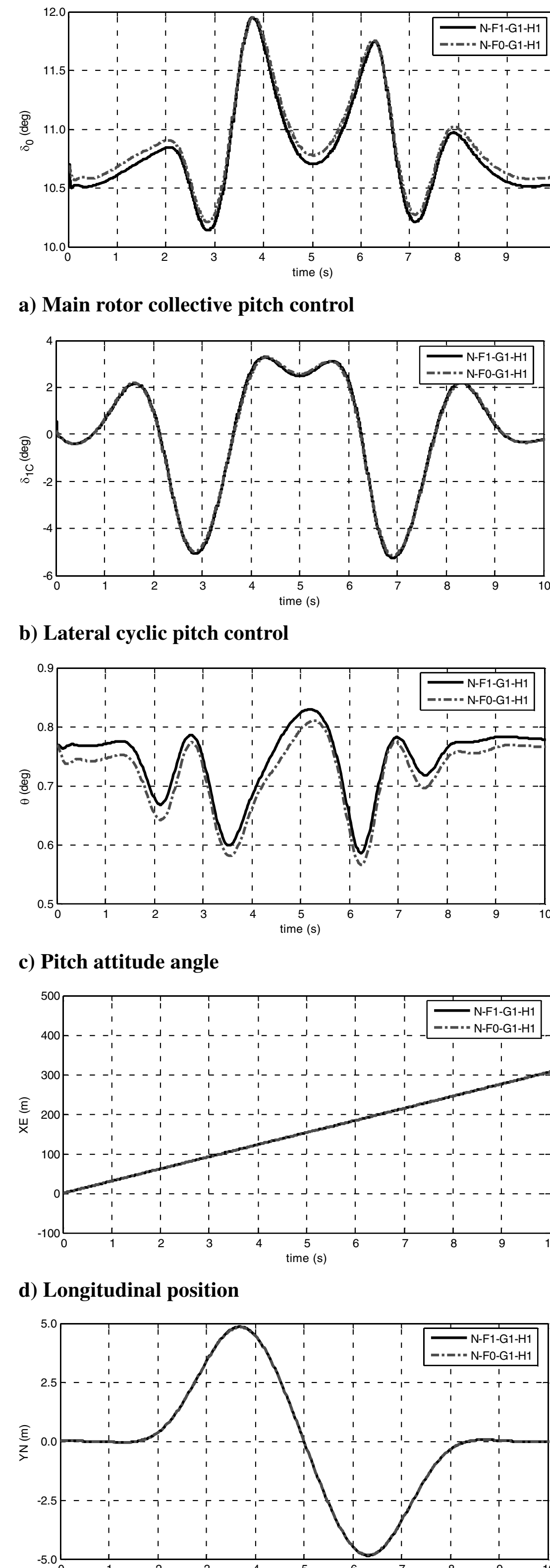
a) Lateral cyclic pitch control

b) Longitudinal cyclic pitch control

c) Pitch attitude angle

d) Lateral position

Fig. 2 Controls and state variations (BFGS formula with different initializations of Hessian matrices for LQ tracking problem).



a) Main rotor collective pitch control

b) Lateral cyclic pitch control

c) Pitch attitude angle

d) Longitudinal position

e) Lateral position

Fig. 3 Control and state variations (SDME method with different state calculation methods).

Various combinations of methods, as defined in Table 1, have been implemented to obtain the converged solutions of the nonlinear optimal control formulation, but only the analyses with the N-F1-G1-H1 and N-F0-G1-H1 methods were convergent. Moreover, any combined analyses with the BFGS update formula failed to achieve a successful solution. Figure 3 compares the converged solutions with the N-F1-G1-H1 and N-F0-G1-H1 methods. Minor deviations in the main rotor collective pitch and pitch attitude angle differentiate the effect of the time integration methods on the solution of the nonlinear system equations. The SDME approach provided performance in integrating the nonlinear motion equation comparable with the Runge–Kutta method.

Conclusions

A new approach to building the Karush–Kuhn–Tucker system has been developed using the state-dependent matrix exponential and its weighted integrals. The proposed method yields better numerical accuracy and faster convergence than conventional methods, the root causes of which are related to the removal of numerical errors of the time integrator and the finite difference formula. Because these errors can be rapidly increasing, especially for a highly nonlinear system with unstable and fast dynamics such as a rotorcraft, the combined approach with the present method and an accurate time integrator for system dynamics is highly recommended for such systems. The ability to provide an accurate initial Hessian matrix could be another advantage of the present approach. Thus, the present method can contribute to the robustness enhancement of the direct-multiple-shooting method.

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